

# Entropy of the Randall-Sundrum black brane world in the brick-wall method

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## Abstract

We calculate the entropy of the brane-world black hole in the Randall-Sundrum(RS) model by using the brick-wall method. The modes along the extra dimension are semi-classically quantized on the extra dimension. The number of modes in the extra dimension is given as a simple form with the help of the RS mass relation, and then the entropy for the scalar modes in the five-dimensional spacetime is described by the two-dimensional area of the black brane world.

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Recently, there has been much interests in the Randall and Sundrum(RS) model to resolve the gauge hierarchy problem [1,2], which is based upon the fact that our universe may be embedded in higher-dimensional spacetimes [3,4]. Furthermore, the various aspects of this model have been studied on the cosmological grounds [5]. In order to study the gauge hierarchy problem, RS have proposed a two-brane model called RS1 model involving a small and curved extra dimension, which is a slice of anti-de Sitter (AdS) spacetime [1]. The negative tension brane is regarded as our universe, and the hierarchy between physical scales naturally appears in our brane. Furthermore, they have studied a single-brane model(RS2 model) by taking  $r_c \rightarrow \infty$ , where  $r_c$  is a radius of the extra dimension [2]. In these models, the nonfactorizable metric is essential as a static anti-de Sitter(AdS) spacetime, which is different from that of the conventional Kaluza-Klein(KK) style in that the extra coordinate is associated with a conformal factor. On the other hand, the static AdS domain wall has been already found as BPS domain walls of supergravity theories in Ref. [6], which is relevant to the RS model in a special case.

Subsequently, it has been shown that the curved domain wall solution in the RS models can be given by transforming the Minkowskian metric by the Schwarzschild one, which is described by a black string solution in five dimensions [7]. The  $d$ -dimensional generalization of this has been shown in Ref. [8] and a brane-world black hole for a rotating case has recently been studied in Ref. [9]. In Refs. [10,11], thermodynamics and aspects of evaporation in brane-world black holes have been studied. To study quantum mechanical aspect of this black brane world (black string intersecting the brane world), we may first consider its entropy, which is expected to satisfy the area law [12]. In this paper, we would like to calculate the statistical entropy of the black brane world in terms of the brick wall method [13–16]. We first derive the brane-world black hole solution directly starting from the equation of motion, and obtain the black brane world system. And then we semi-classically quantize the massive scalar field on this five-dimensional background. First, the modes along the extra dimension is quantized, and then the mass is naturally discretized. Next, we calculate the other modes following the brick wall method, and obtain the desired entropy of

the brane-world black hole by using the RS relation. This shows that the number of modes in the five-dimensional spacetime can be effectively described by the two-dimensional surface.

Let us now consider the RS model in  $(4 + 1)$  dimensions,

$$S_{(5)} = \frac{1}{2\kappa_{(5)}^2} \int d^4x \int dy \sqrt{-g_{(5)}} [R^{(5)} + 12k^2] - \int d^4x [\sqrt{-g_{(+)}}\lambda_{(+)} + \sqrt{-g_{(-)}}\lambda_{(-)}], \quad (1)$$

where  $\lambda_{(+)}$  and  $\lambda_{(-)}$  are tensions of the branes at  $y = 0$  and  $y = \pi r_c$ , respectively,  $12k^2$  is a cosmological constant, and  $\kappa_{(5)}^2 = 8\pi G_N^{(5)}$ . Note that we use  $M, N, K, \dots = 0, 1, \dots, 4$  for  $(4 + 1)$ -dimensional spacetime indices and  $\mu, \nu, \kappa, \dots = 0, 1, \dots, 3$  for branes. We assume orbifold  $S^1/Z_2$  which has a periodicity in the extra coordinate  $y$ , and identify  $-y$  with  $y$ . Two singular points on the orbifold are located at  $y = 0$  and  $y = \pi r_c$ , and two 3-branes are placed at these points, respectively. Note that the metric on each brane is defined as  $g_{\mu\nu}^{(+)} \equiv g_{\mu\nu}^{(5)}(x^\mu, y = 0)$  and  $g_{\mu\nu}^{(-)} \equiv g_{\mu\nu}^{(5)}(x^\mu, y = \pi r_c)$ , respectively.

We now assume the bulk metric as

$$ds_{(5)}^2 = e^{-2k|y|\Phi(x)} g_{\mu\nu}(x) dx^\mu dx^\nu + T^2(x) dy^2, \quad (2)$$

where the moduli field  $T(x)$  is different from  $\Phi(x)$  for the present. From Eq. (1), the equation of motion is given as

$$G_{MN}^{(5)} = T_{MN}^{(5)}. \quad (3)$$

By using the metric (2), the Einstein tensors are calculated as

$$\begin{aligned} G_{\mu\nu}^{(5)} = & G_{\mu\nu} - \frac{1}{T} (\nabla_\mu \nabla_\nu T - g_{\mu\nu} \square T) + 2k|y| (\nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \square \Phi) \\ & - \frac{k|y|}{T} (\nabla_\mu T \nabla_\nu \Phi + \nabla_\mu \Phi \nabla_\nu T + g_{\mu\nu} \nabla_\rho T \nabla^\rho \Phi) \\ & + k^2 |y|^2 (2\nabla_\mu \Phi \nabla_\nu \Phi + g_{\mu\nu} (\nabla \Phi)^2) + \frac{6k\Phi}{T^2} e^{-2k|y|\Phi} g_{\mu\nu} (k\Phi - \delta(y) + \delta(y - \pi r_c)), \end{aligned} \quad (4)$$

$$G_{\mu y}^{(5)} = 3k \left( \partial_\mu \Phi - \frac{\Phi}{T} \partial_\mu T \right) \partial_y |y|, \quad (5)$$

$$G_{yy}^{(5)} = -\frac{1}{2} T^2 e^{2k|y|\Phi} \left( R + 6k|y| \square \Phi - 6k^2 |y|^2 (\nabla \Phi)^2 - \frac{12k^2 \Phi^2}{T^2} e^{-2k|y|\Phi} \right), \quad (6)$$

and the stress-energy tensor is explicitly written as

$$T_{MN}^{(5)} = 6k^2 g_{MN}^{(5)} + \kappa_{(5)}^2 \frac{\sqrt{-g^{(+)}}}{\sqrt{-g^{(5)}}} \lambda_{(+)} \delta(y) g_{\mu\nu}^{(+)} \delta_M^\mu \delta_N^\nu + \kappa_{(5)}^2 \frac{\sqrt{-g^{(-)}}}{\sqrt{-g^{(5)}}} \lambda_{(-)} \delta(y - \pi r_c) g_{\mu\nu}^{(-)} \delta_M^\mu \delta_N^\nu. \quad (7)$$

Since  $(\mu y)$ -component of Eq. (7) vanishes, from Eqs. (3) and (5) we obtain the following relation,

$$G_{\mu y}^{(5)} = 3k \left( \partial_\mu \Phi - \frac{\Phi}{T} \partial_\mu T \right) \partial_y |y| = 0, \quad (8)$$

which yields simply  $\Phi(x) = T(x)$ .

In the  $(\mu\nu)$ -component of Eq. (3), there exist discontinuities resulting from the delta functional source due to the presence of brane tensions at  $y = 0$  and  $y = \pi r_c$ . At this stage, we now consider junction conditions [17], and integrate out the Einstein equation near the branes,

$$\begin{aligned} \int_{0-\epsilon}^{0+\epsilon} dy G_{\mu\nu}^{(5)} &= \int_{0-\epsilon}^{0+\epsilon} dy T_{\mu\nu}^{(5)}, \\ \int_{\pi r_c - \epsilon}^{\pi r_c + \epsilon} dy G_{\mu\nu}^{(5)} &= \int_{\pi r_c - \epsilon}^{\pi r_c + \epsilon} dy T_{\mu\nu}^{(5)}. \end{aligned} \quad (9)$$

The jump along the extra coordinate near the 3-branes gives a relation between brane tensions and the cosmological constant,

$$\lambda_{(+)} = -\lambda_{(-)} = \frac{6}{\kappa_{(5)}^2} k, \quad (10)$$

where we note that the branes at  $y = 0$  and  $y = \pi r_c$  have a positive tension ( $\lambda_{(+)}$ ) and a negative one ( $\lambda_{(-)}$ ), respectively. Using the relation (10) for this brane model, the equation of motion (3) is explicitly given as

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{T} [\nabla_\mu \nabla_\nu T - g_{\mu\nu} \square T] + k^2 y^2 [2 \nabla_\mu T \nabla_\nu T + g_{\mu\nu} (\nabla T)^2] \\ + 2k|y| [\nabla_\mu \nabla_\nu T - g_{\mu\nu} \square T] - \frac{k|y|}{T} [2 \nabla_\mu T \nabla_\nu T + g_{\mu\nu} (\nabla T)^2] = 0, \end{aligned} \quad (11)$$

$$R + 6k|y| \square T - 6k^2 y^2 (\nabla T)^2 = 0. \quad (12)$$

Traces of Eq. (11) and Eq. (12) give the following reduced equations,

$$\begin{aligned}
R + 3k|y|\square T &= 0, \\
\square T - 2k|y|(\nabla T)^2 &= 0.
\end{aligned} \tag{13}$$

In Eq. (13), as a simple constant solution of  $T(x)$ , let us set  $T(x) = 1$ . Then metric solution  $g_{\mu\nu}$  should be determined by  $R = 0$ . Combining  $R = 0$  and  $T(x) = 1$  with Eq. (11), the Ricci flat condition,  $R_{\mu\nu} = 0$ , introduced in Ref. [7], is obtained from the equations of motion. From this condition, it is natural to consider the 4-dimensional Schwarzschild black hole solution as a slice of AdS spacetime as a brane solution,

$$ds^2 = e^{-2k|y|} \left[ - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega_{(2)}^2 \right] + dy^2, \tag{14}$$

where  $d\Omega_{(2)}^2$  is a metric of unit 2-sphere and we set  $G_{(4)} = 1$  for convenience. It is a black string solution intersecting the brane world, which describes a black hole placed on the hypersurface at the fixed extra coordinate. Arnowitt-Deser-Misner(ADM) mass  $\tilde{M}$  of the brane-world black hole measured on the brane is  $\tilde{M} = Me^{-ky_0}$  where  $y_0$  is 0 or  $\pi r_c$ . Then, ADM mass would be exponentially suppressed as  $\tilde{M} = Me^{-ky_c}|_{y_c \rightarrow \infty} \rightarrow 0$  on the negative tension brane for the RS2 model. In our work, we shall focus on the black hole placed at  $y = 0$ .

Let us now study the black hole entropy in terms of the brick wall method. With the help of the RS relation which connects the bulk mass with the mass on the brane, the quantization will be simplified. We now assume the massive scalar field  $f$  on the five-dimensional background (14),

$$(\square_{(5)} - m^2)f = 0, \tag{15}$$

and it is explicitly given as

$$\begin{aligned}
e^{2k|y|} \left[ -\frac{1}{g} \partial_t^2 f + \frac{1}{r^2} \partial_r (r^2 g \partial_r f) + \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta f) + \frac{1}{r^2 \sin^2 \theta} \partial_\phi^2 f \right] \\
+ e^{4k|y|} \partial_y (e^{-4k|y|} \partial_y f) - m^2 f = 0,
\end{aligned} \tag{16}$$

where  $g = g(r) = 1 - \frac{2M}{r}$ . If we set

$$e^{4k|y|}\partial_y(e^{-4k|y|}\partial_y\chi) - m^2\chi + \mu^2 e^{2k|y|}\chi = 0, \quad (17)$$

where  $f(t, r, \theta, \phi, y) \equiv \Psi(t, r, \theta, \phi)\chi(y)$ , then the separation of variables is easily done, and the reduced form of the effective field equation becomes

$$-\frac{1}{g}\partial_t^2\Psi + \frac{1}{r^2}\partial_r(r^2g\partial_r\Psi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Psi) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\Psi - \mu^2\Psi = 0. \quad (18)$$

Note that the above eigenvalue  $\mu^2$  plays a role of the effective mass on the brane. Then further separation of variables is straightforward, which gives

$$\frac{1}{r^2}(r^2g\partial_r R) + \left(\frac{\omega^2}{g} - \frac{\ell(\ell+1)}{r^2} - \mu^2\right)R = 0, \quad (19)$$

where  $\Psi(t, r, \theta, \phi) \equiv R(r)S(\theta)e^{i\alpha\phi - i\omega t}$ . The exact quantization of Eq.(17) seems to be cumbersome. However, in the WKB approximation [13], the wave number  $k_\chi$  of the wave function  $\chi(y)$  is easily written as

$$k_\chi^2(y, \mu) = \mu^2 e^{2k|y|} - m^2. \quad (20)$$

Therefore, the number of modes  $n_\chi$  is obtained as

$$\begin{aligned} \pi n_\chi(\mu) &= \int_0^{\pi r_c} dy k_\chi(y, \mu) \\ &= -\frac{\sqrt{\mu^2 - m^2}}{k} + \frac{m}{k} \tan^{-1} \left[ \frac{\sqrt{\mu^2 - m^2}}{m} \right] \\ &\quad + \frac{\sqrt{\mu^2 e^{2k\pi r_c} - m^2}}{k} - \frac{m}{k} \tan^{-1} \left[ \frac{\sqrt{\mu^2 e^{2k\pi r_c} - m^2}}{m} \right]. \end{aligned} \quad (21)$$

Unfortunately, it is difficult to write the eigenvalue  $\mu$  as a function of  $n_\chi$  since the inverse function is not easily obtained. However, we now consider the RS relation [1], which is key to the breakthrough. It has been used to connect the effective four-dimensional mass with the five-dimensional bulk mass, and it is given by  $\mu = m e^{-ky_c}$  at the branes. We note that in Ref. [1] to resolve the hierarchy problem between physical couplings, the negative tension brane at  $y_c = \pi r_c$  is regarded as our universe. On the other hand, in our black hole case, the positive tension brane has been taken as our spacetime of black brane world since the negative tension brane has naked singularities. Furthermore, at the negative tension brane

the real wave number can not be defined. So, for the present case, it is possible to take  $y_c = 0$  without these problems. Then the RS relation is reduced to

$$\mu = m, \quad (22)$$

which yields

$$\mu = \mu_{n_\chi} = \frac{\pi k}{\gamma} n_\chi \quad (n_\chi = 1, 2, 3, \dots) \quad (23)$$

from Eq. (21) with  $\gamma = \sqrt{e^{2k\pi r_c} - 1} - \tan^{-1} \sqrt{e^{2k\pi r_c} - 1}$ . If we take  $r_c \rightarrow \infty$ , then the mode spectrum is continuous similarly to the quantization on a circle.

At last, the radial wave number  $k_R^2 = g^{-2} [\omega^2 - g \left( \frac{\ell(\ell+1)}{r^2} + \mu^2 \right)]$  from Eq. (19) is semi-classically quantized as

$$\pi n_R(\omega, \ell, \mu) = \int_{2M+h}^L dr g^{-1} \sqrt{\omega^2 - g \left( \frac{\ell(\ell+1)}{r^2} + \mu^2 \right)}, \quad (24)$$

where  $h$  and  $L$  are ultra and infrared cutoffs, respectively which are needed in the brick wall formalism. Now the degeneracy for a given energy is defined by

$$\begin{aligned} g(\omega) &\equiv \pi N = \int d\ell (2\ell+1) \int dn_\chi \pi n_R(\omega, \ell, n_\chi) \\ &= \int d\ell (2\ell+1) \int dn_\chi \int_{2M+h}^L dr g^{-1} \sqrt{\omega^2 - g \left( \frac{\ell(\ell+1)}{r^2} + \lambda^2 n_\chi^2 \right)}, \end{aligned} \quad (25)$$

where  $\lambda = k\pi\gamma^{-1}$ .

The free energy  $F$  for this black brane world system is given as

$$e^{-\beta F} = \sum e^{-\beta\omega} = \prod_{n_R, \ell, n_\chi} \frac{1}{1 - e^{-\beta\omega}}, \quad (26)$$

which is explicitly written as

$$\begin{aligned} \pi\beta F &= - \int_0^\infty d\omega \frac{\beta g(\omega)}{e^{\beta\omega} - 1} \\ &= -\beta \int_0^\infty d\omega (e^{\beta\omega} - 1)^{-1} \int d\ell (2\ell+1) \int dn_\chi \int_{2M+h}^L g^{-1} \sqrt{\omega^2 - g \left( \frac{\ell(\ell+1)}{r^2} + \lambda^2 n_\chi^2 \right)}. \end{aligned} \quad (27)$$

For the reality condition of the free energy, the integration ranges are restricted to  $0 \leq n_\chi \leq \left( \sqrt{\omega^2/g - \ell(\ell+1)/r^2} \right) / \lambda$  and  $0 \leq \ell \leq \left( -1 + \sqrt{1 + 4\omega^2 r^2 / g} \right) / 2$ .

In the approximation of  $L \gg 2M$ , the free energy is evaluated as

$$F \approx -64 \frac{\gamma \zeta(5)}{\pi h^2} \sqrt{\frac{h}{2M}} \left(\frac{M}{\beta}\right)^5 - \frac{3}{8\pi k} \gamma L^3 \int_0^\infty d\omega \frac{\omega^4}{e^{\beta\omega} - 1}. \quad (28)$$

It is interesting to note that Eq. (28) is more or less different from the four-dimensional free energy calculation in the Schwarzschild black hole background [13], which is given as

$$F \approx -\frac{2\pi^3}{45h} \left(\frac{2M}{\beta}\right)^4 - \frac{2}{9\pi} L^3 \int_m^\infty d\omega \frac{(\omega^2 - m^2)^{3/2}}{e^{\beta\omega} - 1}. \quad (29)$$

The reason why the mass term of the scalar field in the bulk spacetime does not appear in the free energy expression (28) on the contrary to the four-dimensional case is that the mass  $m$  in the bulk can be effectively interpreted as that of the four-dimensional mass  $\mu$  in terms of the RS relation (22), and it is integrated out in the free-energy calculation after semi-classical quantization along the extra dimension.

Now the Hawking temperature is defined by

$$T_H = \frac{\kappa_H}{2\pi} \quad (30)$$

where  $\kappa_H^2 = -\frac{1}{2}(\nabla^\mu \chi^\nu)(\nabla_\mu \chi_\nu)|_{r=r_H}$  and  $\chi$  is a time-like Killing vector, which yields

$$T_H = \beta^{-1} = \frac{1}{8\pi M}. \quad (31)$$

Therefore, from the free energy (28) the entropy is

$$\begin{aligned} S &= \beta^2 \left( \frac{\partial F}{\partial \beta} \right) \\ &= \frac{\mathcal{A}_H}{4G_{(4)}} \\ &= 4\pi M^2, \end{aligned} \quad (32)$$

as far as the brick wall is located at  $h = \left(\frac{5\gamma\zeta(5)}{2^8\pi^6}\right)^2 \cdot \frac{1}{2M}$ . Of course, the invariance distance from the horizon is calculated so that in the approximation,  $M \gg h$ ,

$$\begin{aligned} \int_{r=2M}^{r=2M+h} \frac{dr}{\sqrt{1 - \frac{2M}{r}}} &\approx \sqrt{2Mh} \\ &= \left( \frac{5\gamma\zeta(5)}{2^8\pi^6} \right), \end{aligned} \quad (33)$$



which is independent of the black hole mass.

We have calculated the entropy of the massive scalar field in the RS model by using the brick-wall method. In our model, the positive tension brane as a black hole solution was considered in order to avoid curvature singularities whereas in the cosmological consideration the negative tension brane has been identified as our universe to resolve the gauge hierarchy problem.

In our result, it is reminiscent of the holographic principle [18] of black hole physics, so the number of degrees of freedom can be derived by the horizon area of the brane, although this does not mean that the massive modes live on the brane. It would be interesting whether or not the area law of the black hole entropy can be derived from the bulk theory by using the other methods.

A final comments are in order. At first glance, the massless limit does not exist in Eq. (22). However, from the beginning equation (20), one can easily take the massless limit of the scalar field, and the quantized rule is given as  $\pi n_\chi(\mu) = \int_0^{\pi r_c} dy \mu e^{2ky} = \frac{\mu}{2k} (e^{2k\pi r_c} - 1) \approx \frac{\mu}{k} \lambda_0$  where  $\lambda_0 = (e^{2k\pi r_c} - 1)/2$ . As a result, the quantized rule is not changed except the coefficient, that is  $\mu = k\pi/\lambda_0 n_\chi$ . Therefore, the result for the massive case is still valid. The physical significance for the massless limit is that in fact the expected entropy can be obtained without recourse to the RS mass relation (22), while for the massive case the relation is helpful in the derivation of the entropy. At this stage, we do not know physically whether the RS relation (22) is essential or not in deriving the entropy. We hope this problem will be discussed in elsewhere.

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## REFERENCES

- [1] L. Randall and R. Sundrum, *Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett. **83** (1999) 3370 , [[hep-ph/9905221](#)].
- [2] L. Randall and R. Sundrum, *An Alternative to Compactification*, Phys. Rev. Lett. **83** (1999) 4690, [[hep-th/9906064](#)].
- [3] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, *The Hierarchy Problem and New Dimensions at a Millimeter*, Phys. Lett. **B429** (1998) 263, [[hep-ph/9803315](#)].
- [4] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, *New dimensions at a millimeter to a Fermi and Superstrings at a TeV*, Phys. Lett. **B436** (1998) 257, [[hep-ph/9804398](#)].
- [5] P. Binétruy, C. Deffayet, and D. Langlois, *Non-conventional cosmology from a brane-universe*, Nucl. Phys. **B565** (2000) 269, [[hep-th/9905012](#)]; C. Csáki, M. Graesser, L. Randall, and J. Terning, *Cosmology of Brane Models with Radion Stabilization*, [[hep-ph/9911406](#)].
- [6] M. Cvetič, S. Griffies, and S.-J. Rey, *Static Domain Walls in  $N = 1$  Supergravity*, Nucl. Phys. **B381** (1992) 301 ; M. Cvetič, S. Griffies, and H. Soleng, *Local and Global Gravitational Aspects of Domain Wall Space-times*, Phys. Rev. **D48** (1993) 2613, [[gr-qc/9306005](#)].
- [7] A. Chamblin, S. W. Hawking, and H. S. Reall, *Brane-World Black Holes*, Phys. Rev. **D61** (2000) 065007, [[hep-th/9909205](#)].
- [8] W. T. Kim, J. J. Oh, M. K. Oh, and M. S. Yoon, *Brane-World Black Holes in Randall-Sundrum Models*, [[hep-th/0006134](#)].
- [9] M. S. Modgil, S. Panda, and G. Sengupta, *Rotating Brane World Black Holes*, [[hep-th/0104122](#)].

- [10] R. Emparan, G. Horowitz, and R. C. Myers, *Exact Description of Black Holes on Branes*, J. of High Energy Phys. **001** (2000) 07, [[hep-th/9911043](#)].
- [11] R. Emparan, G. Horowitz, and R. C. Myers, *Black Holes Radiate Mainly on the Brane*, Phys. Rev. Lett. **85** (2000) 499, [[hep-th/0003118](#)].
- [12] J. D. Bekenstein, *Black Holes and Entropy*, Phys. Rev. **D7** (1973) 2333 ; S. W. Hawking, *Particle Creation by Black Holes*, Commun. Math. Phys. **43** (1975) 199.
- [13] G. 't Hooft, *On the Quantum Structure of a Black Hole*, Nucl. Phys. **B256** (1985) 727.
- [14] J.-G. Demers, R. Lafrance, and R. C. Myers, *Black hole entropy without brick walls*, Phys. Rev. **D52** (1995) 2245 ; R. B. Mann, L. Tarasov, and A. Zelnikov, *Brick walls for black holes*, Class. Quant. Grav. **9** (1992) 1487; S. Mukohyama and W. Israel, *Black Holes, Brick Walls and The Boulware State*, Phys. Rev. **D58** (1998) 104005, [[gr-qc/9806012](#)]; E. Winstanley, *Renormalized Black Hole Entropy via the 'Brick Wall' method*, [[hep-th/0011196](#)]; E. Abdalla and L. A. Correa-Borbonet, *Black Hole entropy by the Brick Wall method in four-dimensions and five-dimensions with  $U(1)$  charges*, [[hep-th/0012101](#)].
- [15] V. P. Frolov and D. C. Fursaev, *Thermal Fields, Entropy, and Black Holes*, Class. Quant. Grav. **15** (1998) 2041, [[hep-th/9802010](#)].
- [16] S.-W. Kim, W. T. Kim, Y.-J. Park, and H. Shin, *Entropy of the BTZ Black Hole in  $(2+1)$ -Dimensions*, Phys. Lett. **B392** (1997) 311.
- [17] W. Israel, *Singular Hypersurfaces and Thin Shells in General Relativity*, Nuov. Ciment. **B44** (1966) 4349.
- [18] G. 't Hooft, *The Holographic principle: Opening Lecture*, [[hep-th/0003004](#)]; L. Susskind, L. Thorlacius, and J. Uglum, *The Stretched Horizon and Black Hole Complementarity*, Phys. Rev. **D48** (1993) 3743, [[hep-th/9306069](#)]; D. Bigatti and L. Susskind, *TASI Lectures on the Holographic Principle*, [[hep-th/0002044](#)].